Symmetry and tessellation in Islamic art

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Abstract

Islamic visual art is well known for using simple geometrical shapes in infinitely repeatable patterns in a process called tessellation. However, it did not use any type of living thing as tiles (basic units in tessellation). This paper reviews the two major characteristics of Islamic visual art that make it detectable and attractive: symmetry and tessellation, investigating the properties that make them essential in this type of art. First, an overview of Islamic visual art and patterns is presented, discussing some of the basic features of these patterns. Then, an overview of symmetry, its types, and plane symmetry groups is presented, discussing their mathematical properties. And similarly, an overview of tessellation and its types is presented. Finally, the aesthetic preference for symmetry is discussed, considering its psychological effect of it on detecting and memorizing Islamic patterns. It can be concluded that Islamic visual art is special because of its use of symmetry in almost all its artworks.

I. Introduction

Islamic art is defined as "art created by artists whose religion was Islam, for patrons who lived in predominantly Muslim lands or for purposes that are restricted or peculiar to the Muslim population or a Muslim setting," [1] and it is well-known for its use of symmetrical patterns and tessellation in the decoration of religious buildings such as mosques and castles. Symmetry, in simple terms, is a balanced and proportionate similarity between two halves of an object. It implies that one half is the other's mirror image. Tessellation, in addition, is the process of arranging shapes in patterns without leaving any gaps, thus creating a tiling, which is a collection of shaped tiles tilled using symmetrical patterns [2]. This paper represents an overview of geometric symmetry and tessellations in addition to the relation between them (plane symmetry groups) and their application in Islamic art.

II. An Overview of Islamic Art

i. Historical context

When used in reference to art, the word "Islamic" is surrounded by a great deal of confusion. The adjective "Islamic," unlike "Christian," identifies not only a faith but also a whole culture. Therefore, when applied to art, it refers to structures and artefacts created by or for people who lived under Islamic rulers or in social and cultural units that, whether or not they were Muslims themselves, were greatly affected by the Islamic way of life and philosophy. [3]

To fully understand this art, it is crucial to investigate three aspects: first, whether the Arabs who conquered such a large area brought any particular traditions with them; second, whether the new faith imposed any attitudes or rules that demanded or shaped artistic expression; and third, what significant artistic movements the Muslims encountered in the lands they conquered.

Regarding the first aspect, it has usually been believed that the Arabs of Arabia had very little indigenous artistic traditions of any significance, at least in the time just before the Muslim conquest. That is shown clearly in the Ka'ba in Mecca, which was the most important building in the Arabian Peninsula. It was just a nearly cubic building without any architectural or artistic decoration. So, it can be inferred that other buildings were even less impressive. This poverty in artistic development in pre-Islamic Arabia is a result of the paucity of excavations and explorations. [3]

When we turn to those attitudes and requirements that the faith established and which sooner or later influenced Islamic art, there are a huge number of examples. First, the first Masjid was constructed in 622, the year of the Hijra, when the Prophet Muhammad departed Mecca to create the first Islamic state in Madinah. A Masjid (pl. Masajid), or mosque, is the place of worship in Islam. It had to be suitable for the community in area, design, and decoration. There was a huge development in mosque design between different Islamic eras, such as the Umayyads, Abbasids, and Uthmanis. Another example is the Ka'ba in Mecca. It was the qibla and each Muslim is required to pilgrimage to it at least one time in his life if could. The development in Ka'ba and the surrounding mosque, al-Masjid al-Haram, was also huge to suit the increasing number of pilgrims each year. Another viewpoint on this topic is presented in [4]: "The inclusion of these geometrical objects has its roots in the teachings of the Quran that invite people to think over the wisdom lurking in the structure of the cosmos and celestial objects like skies, constellations, stars, the sun, the moon, and the earth itself."

Lastly, when it comes to other civilizations that influenced Islamic art, it is important to mention that all the lands that were taken over by the Muslims in the seventh century, which formed the core of the Islamic empire, had been affected by the classical art of Greece and Rome in its widest sense. Carl Becker stated clearly: "Without Alexander the Great, there would not have been a unified Islamic civilization." [5]

There are many examples of Islamic art, such as domes, mosaics, and mihrabs, and some of those are shown in figures 1, 2, and 3. The first and most popular example is *the Dome of the Rock* in Palestine; the second is *the mosaics of Hisham's Palace* in Palestine too, and the third is *the Great Mosque of Kairouan; Mihrab* in Tunisia. Rather than religious decoration, there are also many types of visual Islamic art. Some of those are calligraphy, earthenware, painting, carpeting, and glassworks.

Figure 1: Dome of the Rock

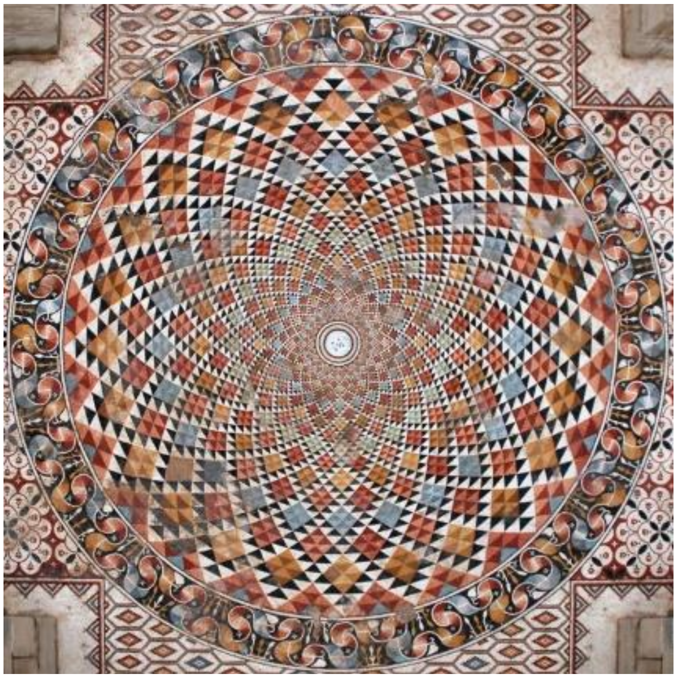

Figure 2: The moasic of Hisham's Palace

Figure 3: Great Mosque of Kairouan; mihrab

ii. Features of Patterns of Islamic Art

Some characteristics of Islamic patterns can be summarized as follows:

1. All patterns, with their different types, consist of smaller units called tiles. Each tile itself consists of simple geometric shapes like squares, circles, triangles, and straight lines.
2. Islamic patterns are infinitely repeatable without any gaps or overlaps. This means that they are not limited to any particular frame. The process of making such patterns is called tessellation.
3. Sometimes, calligraphic art is used with geometric patterns as in *Dome of the Rock* (see figure 1) and some mihrabs such as *mihrab al-Masjid al-Nabawi* (shown in figure 4).
4. Symmetry is also a key element in Islamic patterns. The principles and applications of symmetry are detailed in sections (..and..).
5. Living objects such as animals and plants are not common in Islamic patterns. Instead, Muslim artists found it enough to use only geometric shapes in their designs. "These designs display an amazing variety of geometrical patterns because, according to the Islamic religion, the representation of people, animals, or any real-life objects in art is forbidden." [6]

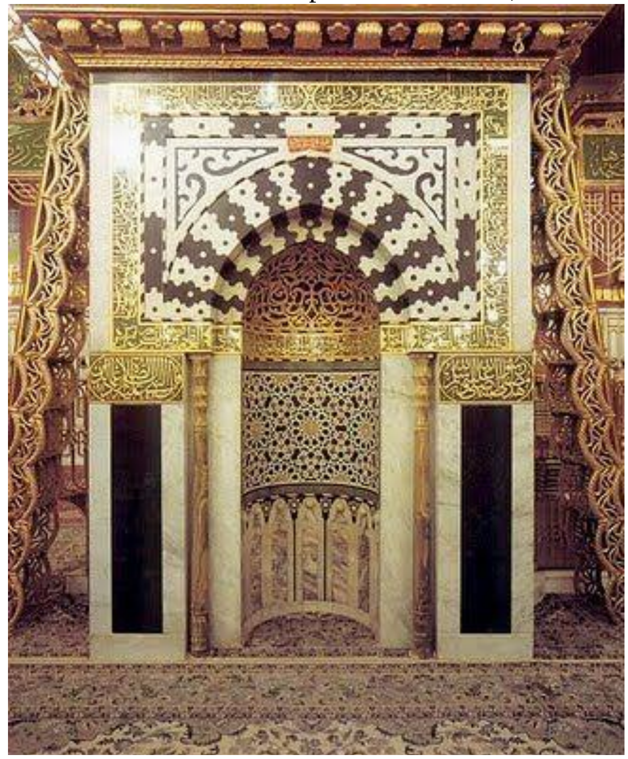

Figure 4: Mihrab al-Masjid al-Nabawi.

The next sections discuss two of these features (2 and 4) in detail investigating the effect of each of them on how popular Islamic art is.

III. Islamic mathematics

Islamic mathematics are the theories that flourished in the areas when Islam was the ruler, mathematics in this period has developed very much. For instance, the decimal notation, and algorithm word came from the name of the great scientist Al-Khwarizmi, which means that he contribute to the concept of algorithms, in addition to completing the square, there are 3 rules that state the relationship between the diameter and the circumstance, the first rule was formulated by Archimedes, and the second rule is in Brahmasphutasiddhanta book, the two rule gives approximate values for π, but Al-Khwarizmi put the third rule which is:

d is the diameter, and p is the periphery

$$p = \frac{62832}{20000} d, or\ \pi\ \approx 3.1416$$

This relation is the most accurate relation and gives the exact value of π. Thabit ibn Qurra discovered that "for integers the sum of two cubes can never be a cube". Abu Kamil also provides rules for manipulating the following algebraic quantities:

$$(a \pm px)(b \pm qx) = ab \pm bpx \pm apx + pqx^2$$

$$(a \pm px)(b \pm qx) = ab \pm bpx \pm aqx - pqx^2$$

$$\sqrt{a.b} = \sqrt{a}\,.\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\sqrt{a}\ \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{ab}}$$

Some impossible problems have been provided solutions for them in this period, for example, the scientist Abu Sahl solved impossible problems, and also explain the construction of a regular heptagon. Al-Samaw'al also has a book called "The Shining Book on Calculation", this book includes very important rules in mathematics, such as:

$$(-ax^n) - (-bx^n) =\ -(ax^n - bx^n), if\ a > b$$
$$(-ax^n) - (-bx^n) =\ +(ax^n - bx^n), if\ a < b$$

The current mathematics is based on Islamic mathematics in a big way, all the mathematics we study now comes from Islamic mathematics, without Islamic mathematics, there will be a large space in the basis of mathematics. [7]

IV. An overview of symmetry

i. Symmetry

As of yet, there isn't a general definition of symmetry on which everyone can agree. Instead, a number of definitions can be discovered from a variety of resources. Despite the fact that the core idea appears to be the same, various words and phrases are frequently used. Additionally, practical definitions are generally based on strong assumptions, such as the notion that geometric symmetries have a Euclidean structure. The majority of time, symmetry is demonstrated rather than defined. [8]

The definition that most fits the purpose of the paper is "A symmetry of some mathematical structure is a transformation of that structure, of a specified kind, that leaves specified properties of the structure unchanged." [9] For instance, a square (the structure), if rotated 90◦ (the transformation), will be the same (the unchanged properties).

As shown in **figure 5**, there are 4 primary types of symmetries for two-dimensional (2D) structures:

1. Translation, which is shifting the plane a certain distance in one direction.
2. Rotation, which is rotating the plane around a fixed point.
3. Reflection, which is plotting the mirror versions of a structure's points.
4. Glide reflection, which is plotting the mirror versions of a structure's points in some fixed line,

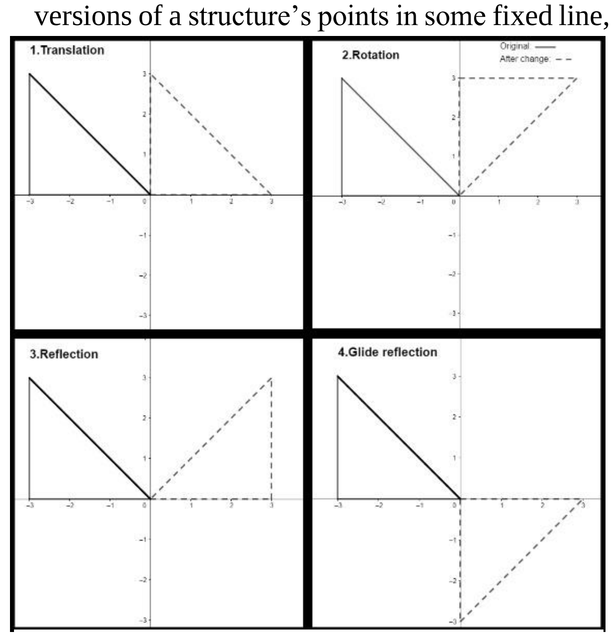

Figure 5: Types of symmetry.

afterwards, translating the result in the direction of that line. [9]

In math terms, if translation occurred for any shape of vertices (x_1, y_1), (x_2, y_2). . . , (x_n, y_n), in the direction of a line $y = mx + c$ that passes through the center of that shape, the new vertices will be $(x_1 + x,\ y_1 + y)$, $(x_2 + x, y_2 + y)$, …, $(x_n + x, y_n + y)$, where x and y satisfy the equation $y = mx + c$, and m is the slope. For the rotation of any shape with vertices (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n) by an angle of θ degrees, the vertices of that shape should be arranged in a matrix as follows:

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix}$$

The next step is to multiply the matrix by the following matrix to give a rotation of θ degrees or radians:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This gives us the following formula:

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The resulting matrix will be the new points of the shape. For the reflection of a given point $A(x_1, y_1)$ about a straight line with a given equation:

$$f(x) = m_1 x + c_1$$

where m is the slope and c_1 is the y-axis intercept, a perpendicular straight line on f with the equation:

$$g(x) = m_2 x + c_2$$

must pass through the two points $A(x_2, y_2)$, $A'(a, b)$ and their mid-point $B(d, e)$, where the mid-point of AA' is the only intersection between f and g . Since g is perpendicular on f, then:

$$m_2 = -\frac{1}{m_1}$$

and

$$g(x) = -\frac{1}{m_1}x + c_2$$

Since the point A is on the straight-line g, then:

$$y_1 = -\frac{1}{m_1}x_1 + c_2$$

Solving for c_2, then:

$$c_2 = y_1 + \frac{1}{m_1}x_1$$

Therefore:

$$g(x) = -\frac{1}{m_1}x + (y_1 + \frac{1}{m_1}x_1)$$

Since:

$$f(x) = g(x)$$

has exactly 1 solution which is $B(d, e)$, therefore:

$$-\frac{1}{m_1}d + \left(y_1 + \frac{1}{m_1}x_1\right) = m_1 d + c_1$$

Solving for d:

$$d = \frac{m_1 y_1 + x_1 - m_1 c_1}{m_1^2 + 1}$$

To find e in $B(d, e)$, d should be plugged in f of g, resulting in:

$$e = \frac{m_1^2 y_1 + m_1 x_1 + c_1}{m_1^2 + 1}$$

Therefore:

$$B\left(\frac{m_1 y_1 + x_1 - m_1 c_1}{m_1^2 + 1}, \frac{m_1^2 y_1 + m_1 x_1 + c_1}{m_1^2 + 1}\right) = \left(\frac{x_1 + a}{2}, \frac{y_1 + b}{2}\right)$$

Therefore:

$$\frac{m_1 y_1 + x_1 - m_1 c_1}{m_1^2 + 1} = \frac{x_1 + a}{2}$$

And

$$\frac{m_1^2 y_1 + m_1 x_1 + c_1}{m_1^2 + 1} = \frac{y_1 + b}{2}$$

Therefore:

$$a = \frac{2m_1 y_1 + x_1(1 - m_1^2) - 2m_1 c_1}{m_1^2 + 1}$$

And

$$b = \frac{y_1(m_1^2 - 1) + 2m_1 x_1 + 2c_1}{m_1^2 + 1}$$

Therefore:

$$A'\left(\frac{2m_1 y_1 + x_1(1 - m_1^2) - 2m_1 c_1}{m_1^2 + 1}, \frac{y_1(m_1^2 - 1) + 2m_1 x_1 + 2c_1}{m_1^2 + 1}\right)$$

Applying this formula to all the points of a shape with vertices $(x_1,\ y_1)$, (x_2, y_2), ..., $(x_n,\ y_n)$, it will be reflected about a straight line with a given equation:

$$y = m_1 x + c_1$$

For glide reflection of a shape with vertices $(x_1,\ y_1)$, (x_2, y_2), ..., $(x_n,\ y_n)$, the reflection formula about a straight line with the equation of

$$y = mx + c$$

should be applied. Afterwards, translation formula in the direction of the same straight line should be applied, resulting in glide reflection.

Geometric patterns are often analyzed based on their symmetry properties. This gives a strong connection between geometry and algebra as the collection of symmetry operations on a figure or a pattern constitutes a group (which will be discussed later in this section). Islamic patterns often cover enormous areas and recur in two non-parallel ways (walls and ceilings). They have translational symmetries (which was mentioned in the previous topic) in nonparallel directions since many of them are easily imaginable as continuing forever. Additionally, they might or might not have rotational, reflective, or glide-reflective symmetries (which were mentioned in the previous topic).[10]

A pattern's symmetry group can be used to describe any pattern with translational symmetry in two non-parallel directions. A remarkable finding is that, after looking into every possibility, it can be demonstrated that there are exactly 17 such groups up to isomorphism (patterns). These are frequently referred to as the wallpaper groupings. [10] The names of the 17 plane symmetry groups are: $p4m$, $p6m$, cmm, $p4g$, $p6$, pmm, $p4$, $p31m$, $p2$, $p3$, pm, pmg, $p3m1$, cm, $p1m$, pgg and pg, but there are only 5 groups which appear frequently in Islamic art, which are pmm, cmm, $p4m$, $p6m$ and $p6$. [11]. In order to understand these groups, tessellation and tilling should be first defined.

V. An overview of tessellation

i. Tessellation

Tessellation is simply collecting small 2-d geometric shapes in a specific pattern, to make beautiful shapes that can impress the human eye. Tessellation can be made by regular polygons (square, equilateral triangle, and a hexagon), rectangles, rhombus, trapezoids, and isosceles, determining if the shape can be tessellated or not just depends on the gaps between the shapes when put next to each other, thus, because of the gaps, the circle and pentagon cannot be tessellated, and the octagon has limited options to be tessellated. If you only want to use one regular polygon to make a tessellation, there are only three possible polygons to use: triangle, square, and hexagon. Tessellation can be made by starting with one or more figures, rotating, translating, reflecting them, or using a mix of transformations to create a pattern that repeats. [12] Star polygons do not naturally tessellate. However, star polygons can be combined with other polygons to create a wide range of stunning tessellations. Islamic art is renowned for its intricate tessellations, many of which prominently incorporate star polygons. [13]

ii. Types of tessellations

1. Regular tessellations:
   Regular tessellations are tiling patterns that are created by arranging just one form in a specific pattern. Regular tessellations can be divided into three categories: triangles, squares, and hexagons.
2. Semi-Regular Tessellations:
   A semi-regular tessellation results from the sharing of a vertex by two or three different types of polygons.

3. Demi-Regular Tessellations:
   Combinations of two or three different polygonal configurations.
4. Irregular Tessellations:
   An irregular tessellation differs from a semi-regular tessellation in that it is not constructed of regular polygons. An endless number of irregular tessellations can be made with these irregular figures.
5. Monohedral Tiling:
   Monohedral tiling creates patterns using just one shape that rotates or flips. Congruent is the mathematical term for such a shape. Shapes with three and four sides tessellate at least in one direction.
6. Duals:
   By using the center of each polygon as the vertex and connecting the centers of nearby polygons, you can create a dual of a standard tessellation. Hexagonal and triangular tessellations are duals of each other, while a square tessellation is its own dual.

iii. Tessellation Transpositions

1. Translation:
   A tessellation where the pattern is repeated by sliding or moving. Observe how the repeated images move from side to side, up and down, and in both directions.
2. Rotation:
   A tessellation where the pattern is repeated by spinning or rotating. Take note of how the images rotate as they repeatedly repeat.
3. Reflection:
   A tessellation where the shape reflects or flips to repeat.

Observe how the repeated pictures reflect one another. [12]

VI. Relation between symmetry and tessellation

If an infinite tessellation can be moved while maintaining its original form, that tessellation is said to have translational symmetry (perfectly overlie itself). A periodic tessellation is one that has translational symmetry.

If the entire tessellation can be rotated by 1/n of a full revolution about a point and stay intact, the tessellation is said to have n-fold rotational symmetry at that point (where n is an integer).

If the entire tessellation can be translated along a line, reflected about that line, and stay unaltered, the tessellation is said to have glide reflection symmetry. [13]

i. Plane symmetry groups explanation and examples form Islamic art

To understand the frequently used plane symmetry groups in Islamic art, specific notation should be used. The symbol T is used to identify a tile on which a motif should be placed. In addition, it is assumed that the template motif has no symmetries. T will be referred to as a template tile and motif as a template motif. To reflect T about its horizontal edge or vertical edge, it should be written as T_H and T_V respectively. To reflect it about a line $\overline{XY}$ for example, it will be written as $T_{\overline{XY}}$. To rotate T upside down, it will be written as T_π, and to rotate it about a point M with an angle of θ, it will be written as T_θ^M. In addition, the $+$ in $T + T_\pi$ is used to denote the operation of sticking tiles together (in this example it means T sticking with its inverted version). Finally, U is a unit tile. For instance, $U = T + T_\pi$ means that the unit tile is T sticking to its inverted version. The unit tile can be then translated several times (by using the translation formula) to obtain a wallpaper group. [11] Now, the 5 plane symmetry groups will be explained using the notation above with providing figures for illustration.

1. pmm (Figure 6a&6b):

$$T = \text{Any rectangle}$$

$$U = T = T_V + T_\pi + T_H$$

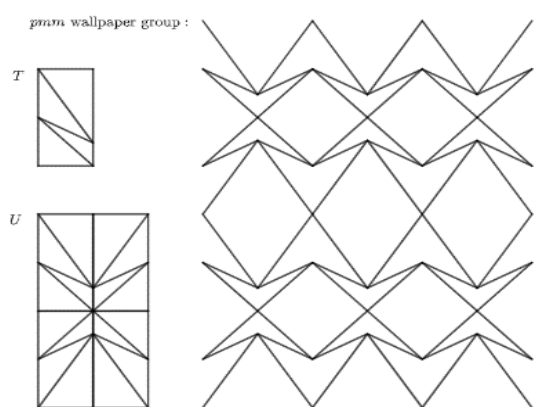

Figure 6a: An Illustration of pmm wallpaper group

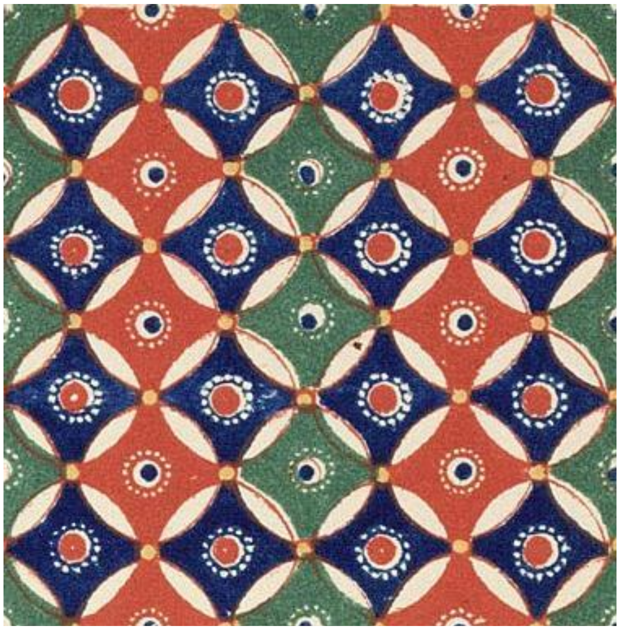

Figure 6b: An Egyptian pmm pattern

2. cmm (2 methods) (Figure 7a&7b).

Method 1:

$$T = \text{Any right-angled triangle}$$

$$U = T = T_V + T_\pi + T_H$$

Method 2:

$$T = \text{Any rectangle}$$

$$U = T + T_\pi + T_H + T_V + T_\pi + T + T_V + T_H$$

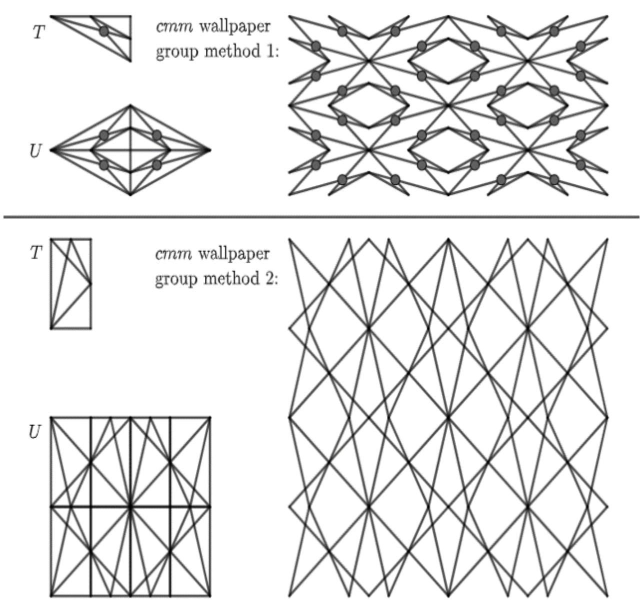

Figure 7a: An Illustration of cmm wallpaper group

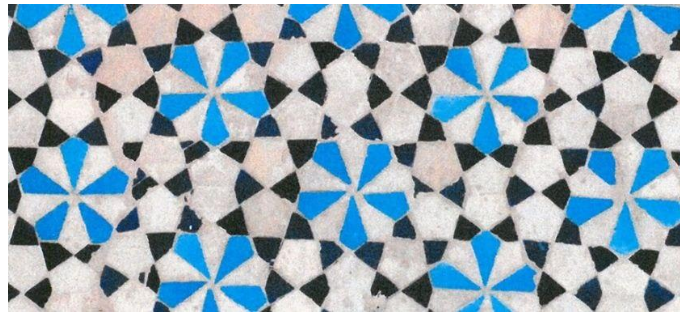

Figure 7b: An Iranian cmm pattern

3. $p4m$ (Figure 8a&8b):

$$T = \text{Any right-angled-isosceles triangle}$$

$$T_1 = T + T_{\overline{XY}}$$

$$U = T_1 + T_{1_{90}}^{Y} + T_{1_{180}}^{Y} + T_{1_{270}}^{Y}$$

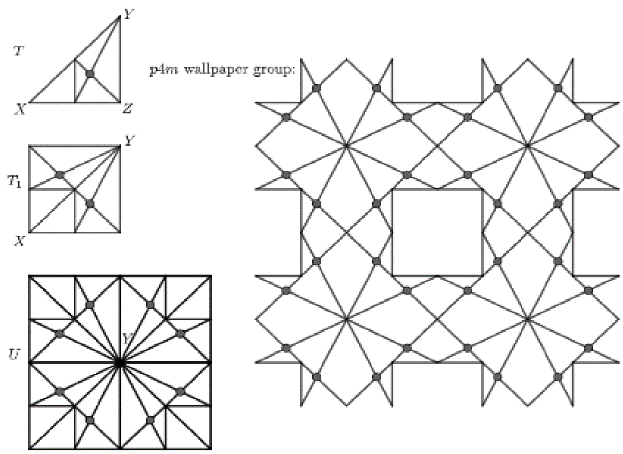

Figure 8a: An Illustration of p4m wallpaper group

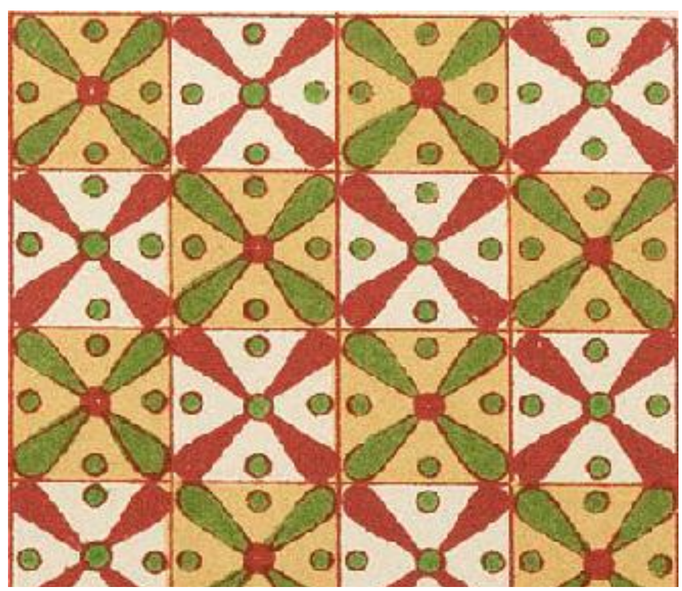

Figure 8b: An Egyptian p4m pattern

4. $p6m$ (Figure 9a&9b):

$$T = \text{Any 30, 60, 90 triangle}$$

$$T_1 = T + T_V$$

$$T_2 = T_1 + T^M_{1_{120}} + T^M_{1_{240}}$$

$$U = T_2 + T_{2_\pi}$$

$$\text{Or } U_2 = U + U^F_{120} + U^F_{240}$$

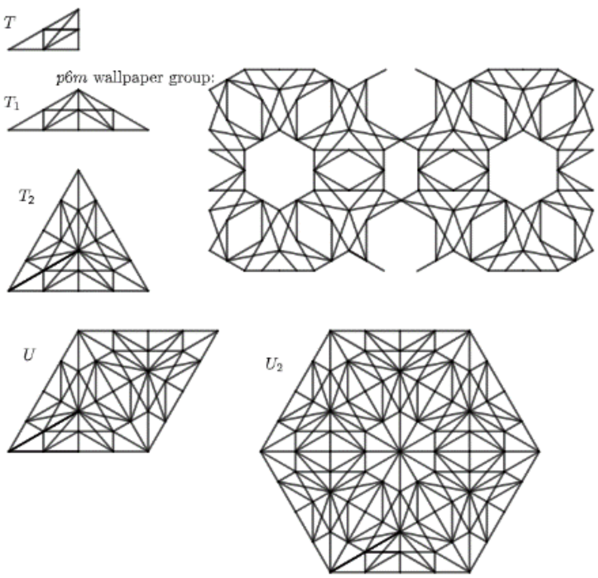

Figure 9a: An Illustration of p6m wallpaper group.

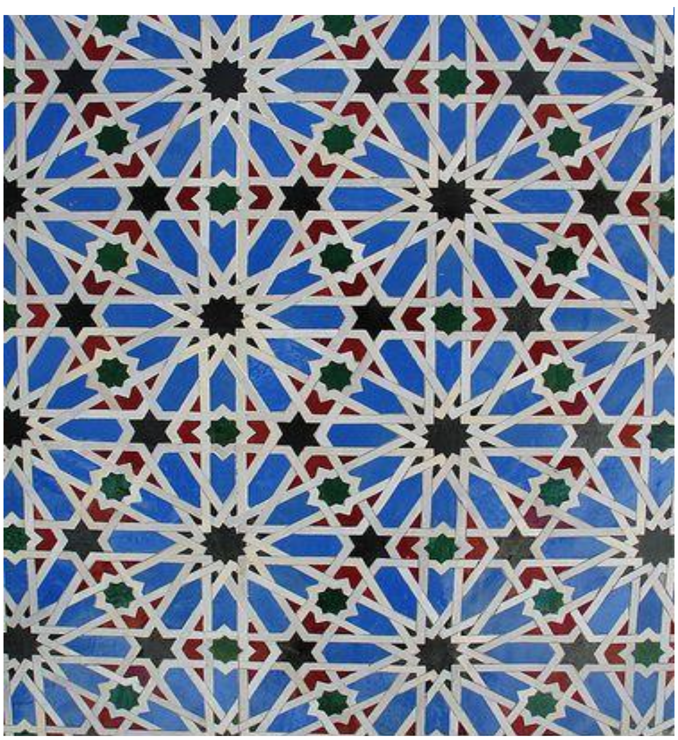

Figure 9b: A Spanish p6m pattern.

5. $p6$ (Figure 10a&10b):

$$T = \text{Any 30-angled-isosceles triangle}$$

$$T_1 = T + T^M_{120} + T^M_{240}$$

$$U = T_1 + T_{1_\pi}$$

Or

$$U_2 = U + U^F_{120} + U^F_{240}$$

[11]

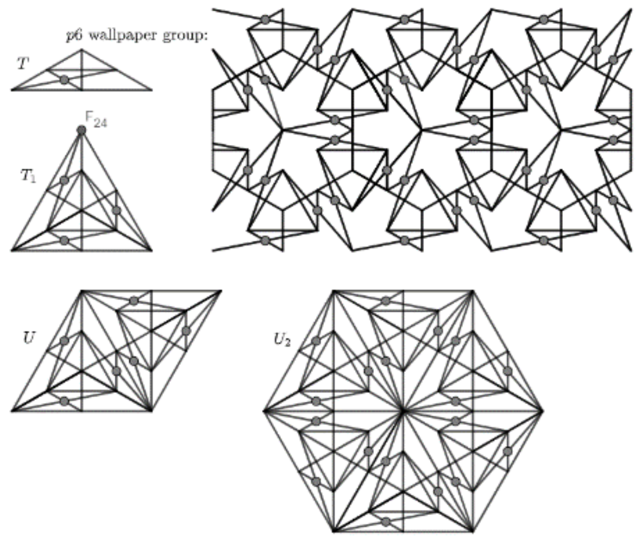

Figure 10a: An Illustration of p6m wallpaper group.

Figure 10b: An Arabian p6 pattern.

VII. Symmetry as an aesthetic factor

People have a universal aesthetic preference for symmetry. Symmetry is considered as a "law of beauty." As Herman Wey wrote in his classic book, "Symmetry," "beauty is bound up with symmetry." [14] This concept of beauty or perfection was assumed to apply in both nature and art. It was thought that every species or natural kind has an ideal set of proportions, an ideal symmetry, from which all individuals deviate to a greater or lesser degree, but by their approximation to which their beauty can be assessed. The objective of the artist was to reproduce, not the imperfect natural things that served as his models, but the ideal symmetry unique to their species. The amount to which the artist was deemed to have embodied this ideal symmetry in his work more precisely than it was represented in his models was used to determine the level of his achievement. [15] So, seeking perfection, Muslim artists used the principles of symmetry in almost all their artwork.

Symmetry preferences may be shown in a variety of artistic forms, including painting, sculpture, architecture, and music. [16] It has been extensively investigated in adults, kids, and infants using a variety of study methodologies, from behavioral psychology to neuroscience. Adults recall symmetrical visual presentations more effectively and more quickly than asymmetrical displays. [17]

Ca´rdenas and L. Harris [18] conducted three experiments. In Experiments 1 and 2, they presented images of pairs of faces to undergraduate students and asked them to select the more attractive face in each pair. The images showed faces that were physically symmetrical and asymmetrical and had been adorned with either symmetrical or asymmetrical decorations, as was common in many preindustrial communities. These two studies found that individuals regard faces with more symmetry as being more attractive and that symmetrical facial ornamentation enhances facial attractiveness. In Experiment 3, students made comparable decisions between pairs of abstract designs that varied in terms of form, coloring, and feature orientation and were drawn from other cultures. According to the results of this study, abstract art patterns have the same impact as faces. And because using abstract art is one of the distinguishing features of Islamic art, it can be concluded that Islamic patterns have similar effect.

VIII. Conclusion

To sum up, the paper discussed two of the most critical characterizations of Islamic visual art, symmetry and tessellation, from a mathematical viewpoint. First, a historical overview of Islamic art in its early beginnings was introduced, pointing out some crucial features of Islamic art. Secondly, to form the basis of the mathematical view, an overview of Islamic mathematics was included. Then, symmetry and tessellation, with their mathematical properties, were discussed. The relationship between the two concepts (plane symmetry groups) was then discussed with its applications in Islamic visual arts. Finally, a psychological viewpoint on symmetry was introduced, showing how symmetry in Islamic patterns affects our brain. The significance of this viewpoint is to recognize how Islamic artists used plane geometry principles in a very aesthetic way.

IX. **References**

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