Quaternions and Octonions review, relationship with Cayley Dickson Construction, and Contribution to Quantum Mechanics Interpretations and 3D Rotation

Mazen Hisham, STEM High School for Boys - 6th of October.

Mariam Manaa, Gharbiya STEM School.

Mentor: Hazem Abdel-Mughni, STEM High School for Boys - 6th of October.

Abstract

Mathematics has always been the mother of sciences. The main reasons behind this are the broadness of mathematics and its compelling ability to translate theory into laws and algorithms to help us understand the universe better. The discovery of imaginary numbers was a critical moment in the history of mathematics, extending its horizon by solving undefinable polynomials with such a revolutionary idea. This paper aims to clear the common misconception about the existence of a finite number of numerical systems, explain their applications, and extend basic algebraic properties to conclude their origin. The focus of this paper is on the abstract mathematical approach to higher-dimensional complex systems, or hyper-complex number systems, of Quaternions and Octonions, discussing the historical background of these systems, the related fundamental algebraic concepts, their construction, properties, operations, and finally their real-life applications. Hyper-complex number systems are not only beneficial in computer science and theoretical physics but also groundbreaking within the fields of mathematics. Accordingly, this paper summarizes the findings throughout the history of hyper-complex numbers and demonstrates their ability to be applied in physics, quantum mechanics, computer graphics, and more.

I. Introduction

Hypercomplex numbers, one of the most significant contributions to the field of mathematics, are a generalization of complex numbers and extensions to the widely known two-dimensional complex systems [1]. The development of hypercomplex numbers had a long accumulative base of algebraic concepts that mathematicians have built throughout the centuries, from Ancient Greek mathematics that introduced the fundamental ideas of imaginary numbers in the sixteenth century [2]. The geometrical representation of the complex plane that consists of a real axis and an imaginary axis was introduced by Carl Friedrich Gauss, providing the ability to express complex numbers as ordered pairs. After multiple trials to extend the two-dimensional complex system to higher dimensions aiming to proceed with modeling three-dimensional rotations [3], Irish Mathematician William R. Hamilton constructed a four-dimensional complex system that represents numbers on the form [3]:

$$q = a + bi + cj + dk \quad (1.1)$$

He named these sets the Quaternions – a group of four things in Latin – the first hypercomplex numbers [4], [3].

The new number system introduced by Hamilton was a crucial transition in the world of algebra and correspondingly in the world of mathematics in the nineteenth century alongside non-Euclidian geometry since the Quaternion number system broke the traditional rules of vector algebra. For instance, quaternions do not have the property of commutativity under multiplication [3].

$$ij = -ji \quad (1.2)$$

Thus, this discovery has opened new windows in algebra and vector analysis.

A normed division algebra is an algebra A and a vector space with $\|ab\| = \|a\|\|b\|$ [7]. Hamilton's discovery is now considered the third of four known normed division algebras: Real Numbers $\mathbb{R}$, Complex Numbers $\mathbb{C}$, Quaternions $\mathbb{H}$, and Octonions $\mathbb{O}$. Octonions, the fourth normed division algebra, was discovered after quaternions by a colleague of William R. Hamilton named John T. Graves. Like quaternions, Octonions are eight–dimensional number systems that form a new non-associative and non-commutative algebra [7].

Hypercomplex numbers have shown their importance in various fields of theoretical physics and engineering, such as the compelling contributions of quaternion algebra in face recognition and robot kinematics. Additionally, quaternions led to forming the basics of the modern theory of relativity [8]. Accordingly, the broad topic of hypercomplex number systems is worthy of investigation throughout this paper due to its significant additions to modern technology and its beneficial connections to other branches of mathematics.

II. Groundwork: Algebraic Concepts

i. Elementary Definitions:

To set off the journey of the hyper-complex numbers, it is essential to construct some elementary definitions. According to the elementary algebra the real numbers $\mathbb{R}$ is the set of all real values and they are represented as a one-dimensional line. The complex numbers $\mathbb{C}$ were formulated depending on the i or in simple terms the imaginary number [9, 10, 11].

$$i = \sqrt{-1} \quad (2.1)$$

The complex numbers are two-dimensional numbers and are in the form of

$$z = a + bi \quad (2.2)$$

Were a, b $\in \mathbb{R}$. Each complex number consists of a real part "a" and imaginary part "bi" [9, 10, 11].

ii. Abstract Definitions:

After dealing with some elementary high school concepts, it is time to introduce the required abstract concepts to start our journey. While dealing with the hyper-complex numbers, vector spaces will be finite-dimensional modules of over $\mathbb{R}$ [7, 13, 14, 15, 16]. Vector space is a set V whose elements are called "Vectors," generalizing the concept, vector spaces are "commutative groups" under addition. Nevertheless, vector spaces are even further than commutative groups. Vector spaces can be scaled [13, 14, 15, 16].

$$\vec{V} = (v_1, v_2, v_3, \dots, v_n) \ \& \ c \in \mathbb{R} \quad (2.3)$$

$$c \cdot \vec{V} = (c \cdot v_1, c \cdot v_2, c \cdot v_3, \dots, c \cdot v_n) \quad (2.4)$$

"c" is called a scalar. Scalars are considered as fields F. Thus, $v \in V$ is a vector and $f \in F$ is a scalar $\rightarrow$ $f \cdot v \in V$ (a "scaled vector") [14, 15, 16].

An algebra A will be a vector space that is equipped with a bilinear map (a function combining elements of two vector spaces to yield an element of a third vector space), $m: A \times A \rightarrow A$, this property is called "multiplication" that is abbreviated as m [7, 13, 16, 17]. There is an element $1 \in A$ such that $m(1, a) =$

$m(a, 1) = a$. The operation called multiplication can be abbreviated as $m(1, a) = ab$ [7, 14]. Since we are dealing with abstract concepts in algebra, we do not assume that our algebras are associative. In algebra A if every non-zero element has an inverse and if the operations of left and right multiplication by any non-zero element are reversible, then A is called a skew field, which is also called a division algebra ring when A is finite-dimensional over k [7, 13, 14, 16].

A normed division algebra is also an algebra A and a normed vector space that has $\|ab\| = \|a\|\|b\|$. Therefore, A is a division algebra. We must say that algebra A has multiplicative inverses, such that for every non-zero $a \in A$ exists $a^{-1} \in A$ which satisfies $aa^{-1} = a^{-1}a = 1\ \forall\ A$ An associative algebra has multiplicative inverses ⟺ it is a division algebra [7, 14, 16].

The associativity of an Algebra can be ranked to three levels of associativity. An algebra A is power-associative if the subalgebra created by any one element is associative [7, 18, 21].

$$a\big(a(aa)\big) = \big(a(aa)\big)a = (aa)(aa) \quad (2.5)$$

It is alternative if the subalgebra generated by any two elements is associative [7, 19, 20].

$$a(ab) = b(aa) = (ba)a = (aa)b \quad (2.6)$$

In conclusion, if the subalgebra generated by any three elements is associative, the algebra is associative [7], [17], [1].

$$c(ab) = (ca)b = a(bc) \quad (2.7)$$

Any algebra has a trilinear map in form of $m: A \times A \times A \rightarrow A$. This trilinear map is called the associator. The associator is in the form of $(a, b, c) = (ab)c - a(bc)$. The associator is a formula that measures the failure of associativity like the commutator $(a, b) = ab - ba$ that measure the failure of commutativity. Hence, we can conclude that if $(a, b, c) = 0$, then the algebra is associative [7, 17].

Theorem 1: The Real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ are the only normed division algebras. Moreover, The Real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ are the only alternative division algebras. Additionally, all division algebras have dimension 1, 2, 4, or 8.

The previous theorem was likely a combination of three theorems to relate and generalize the properties of the Real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. The concept of the $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ being the only normed division algebras was discovered by Hurwitz in 1898 [7, 22]. The concept developed over the years till the year 1930, when Zorn came up with his theorem that the $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are also the only alternative division algebras [7, 20]. After that, Kervaire [7, 23] and Bott–Milnor [7, 24] have proved that all the division algebras have 1, 2, 4, or 8 dimensions independently.

III. Historical Exploration Through Higher-dimensional Complex Numbers

The ancient Greeks claimed to be the first "true" mathematicians to think of numbers as quantities for measurement, not as something abstract. Accordingly, Mathematics back then was best described as 'The Science of Quantities": Lengths, areas, volumes, etc. [25]. Nevertheless, this idea did not hold true for long. The Pythagorean theorem that became widely known by the fifth century BCE led to the unveiling of the existence of irrational numbers, as it was found around 430 BCE that the lengths of the diagonals of the squares were not expressible as finite portions of the unit (i.e., the square root of two is irrational) [26]. Henceforth, the realm of mathematics has been expanding abstractly, from discovering the negative numbers in China [27] to the introduction of the idea of the number zero and the production of a new algebra in the Islamic world by mathematician Muhammed ibn Musa al-Khwarizmi (780-850) [28], and to the discovery of imaginary numbers in the sixteenth century [29].

i. The History of Complex Numbers

A cubic equation associated with a problem on Arithmetica by Diophantus (AD 200-AD 284) was as follows:

$$x^3 + x = 4x^2 + 4 \quad (3.1)$$

It was noy known how the solution was determined to be 4, but it was expected that Diophantus simplified the equation to the form:

$$x(x^2 + 1) = 4(x^2 + 1) \quad (3.2)$$

The value of x as 4 can satisfy this equation, but the solutions to similar special cubic remained a questionable manner. Although Fra Luca Pacioli (1447-1517) stated in his Summa de Arithmetica, Geometria, Proportioni, et Proportionalita that there's no solution for such cubic. several mathematicians, especially Italian scholars, nevertheless, insisted on making attempts to find a solution [2]. Scipione del Ferro (1465-1526), between 1500 and 1515, found an algebraic method to solve cubic equations of the form:

$$x^3 + cx = d \quad (3.3)$$

Del Ferro kept his method a secret, but he gave it to his student Antonio Maria Fiore (First half of the sixteenth century) prior to his death. Despite the fact that he didn't publish the solution, mathematician Niccolo Tartaglia (1500-1557) claimed to find a solution for the cubic

$$x^3 + bx^2 = d \quad (3.4)$$

Consequently, Fiore challenged him in a thirty-problem contest featuring different cubic cases. Tartaglia won, discovering the solution in 1535 [30]. Girolamo Cardano (1501-1576), who heard of the contest and Tartaglia's solution, wanted to add the key, under the name of Tartaglia, to the new textbook he was working on. After accepting Cardano's invitation to Milan, Tartaglia released his solution to Cardano under the oath of not publishing it. Later, Cardano discovered that the breakthrough of finding the cubic formula was Del Ferro's work in the first place. Accordingly, he gave himself the right to break his oath with Tartaglia and publish the solution to the cubic in his Ars Magna (The Great Art) in 1545 [31].

Cardano's Ars Magna featured one case of a cubic equation where $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ were solutions to the quadratic $x(10 - x) = 40$. He described the square root of a negative number as "mental torture" and proceeded with multiplying both solutions to get $25 - (-15)$, which is equal to 40, solving the equation. In 1572, Rafael Bombelli (1526-1572) was the first mathematician accepting the existence of this "mental torture" (i.e., imaginary numbers) and concluded in his Algebra that real numbers can be originated from imaginary numbers [32].

The development of this controversial idea has gone through many stages ever since. For instance, René Descartes (1596-1650) came up with the term "imaginary," providing further geometrical explanations that the imaginary slope *i* is $\frac{0}{0}$, which is indeterminant, making it impossible to form a geometrical construction of imaginary numbers [33]. As the series of mathematicians who tried to investigate this idea continued their work, it's worthy of note that Leonhard Euler (1707-1783) introduced the symbol *i* with a value of $\sqrt{-1}$ [34]. Moreover, he showed that complex roots occur in conjugate pairs: if a polynomial has a root of $a + b\sqrt{-1}$, another root $a - b\sqrt{-1}$ must exist [35].

The geometrical representation of Complex numbers on the form $ax + bi$ that we know today as "The Complex Plane" is accredited to Carl Friedrich Gauss (1777-1855) in the nineteenth century, and hence the complex plane is referred to as the "Gaussian Plane" in his honor, and the term "Complex" is also his.

Thanks to Gauss, the concept of imaginary numbers became widely accepted by mathematicians, after a long history of calling them "impossible" and "intolerable" [36].

ii. The History of Quaternions

The leading character of this section, William Rowan Hamilton (1805-1865), was able to construct complex numbers from real numbers, complementing the work of fellow mathematicians, namely Augustus De Morgan (1806-1871) and George Peacock (1791-1858), who aimed for justifying the use of harmful and complex numbers. Hamilton studied the operations of complex numbers in the two-dimensional plane and the geometrical interpretations of these operations. As a physicist, he knew how necessary it is in physics to involve problems in three-dimensional spaces. He suggested that it must be possible to develop a system of such operations in three dimensions and even n dimensions [37]. Accordingly, Hamilton was looking for numbers that hold for the following properties:

Proposition 1:

1. Associativity holds for multiplication and division.
2. Commutativity holds for addition and multiplication.
3. It is distributive.
4. Division is unambiguous
5. Numbers obey the law of moduli; if $(\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3})$ $(\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3})$ = $(\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3})$, then $(\boldsymbol{a_1^2}, \boldsymbol{a_2^2}, \boldsymbol{a_3^2})$ $(\boldsymbol{b_1^2}, \boldsymbol{b_2^2}, \boldsymbol{b_3^2})$ = $(\boldsymbol{c_1^2}, \boldsymbol{c_2^2}, \boldsymbol{c_3^2})$

The triplets which Hamilton tried to construct were of the form:

$$a + bi + cj \quad (3.5)$$

where j is the new imaginary unit, $j^2 = -1$, and the plane consisted of three mutually perpendicular axes: the real axis, the i-axis, and the j-axis. The significant problem he faced was multiplyings his triplets. In his letter to John T. Graves (1806-1870), whose enthusiasm encouraged Hamilton to work on the theory of triplets, Hamilton discussed this problem stating that if two of his triplets, $a + bi + cj$ and $x + yi + zj$, the product is supposed to equal:

$$ax - by - cz + i(ay + bx) + j(az + cx) + ij(bz + cy) \quad (3.6)$$

The problem arose from *ij*: if multiplying by i is geometrically a rotation about the *j* – axis in the 3 – dimensional plane, then *ij* is just the same as j – rotating about itself and vice-versa for multiplying by *j*, and both lead nowhere. Throughout his attempts to find a solution, he assumed that *ij=1* or *ij=-1* so that the square of *ij* will be equal to 1, but neither of this assumption held true for the law of moduli. He cared about his numbers to hold true for the law of moduli that he didn't mind neglecting the axioms of associativity and commutativity in the field. By assuming that $ij = 0$, the product seems to hold true for the law of moduli, but the product of *ij* itself is in violation of the rule since the modulus of both *i* and *j* is 1 instead of 0. The same suppression of the term can be obtained by assuming that the $ji = -ij$ and that $ji = k$, $-ij = -k$. By multiplying the previously mentioned $a + bi + cj$ and $x + yi + zj$, the result will be as follows:

$$ax - 2b - 2c + i(a + x)b + j(a + x)c + k(bc - bc) \quad (3.7)$$

This results in the suppression of the coefficient k as desired and still finds the product-point. Hamilton then found that adding a fourth dimension to his triplets plane will solve the algebraic problem in multiplication, adding a new imaginary unit k equal to the product of *i* and *j*, where

$$\begin{cases} i^2 = j^2 = k^2 = -1 \;, \quad jk = -kj = i \\ ij = -ji = k \quad\;, \quad ki = -ik = j \end{cases} \quad (3.8)$$

The new extended complex system lost the axiom of commutativity, as it was mentioned that $ji = -ij$. Hamilton spent 13 years from 1830 to 1843 trying to figure this problem out, and finally wrote the preceeding attempts and his final conclusions on

October 16th 1843, to Graves, introducing his new theory of quadruplets or Quaternions: Numbers on the form

$$q = a + bi + cj + dk \quad (3.9)$$

Where a, b, c, and d are real numbers and i, j, and k are imaginary units, with the fundamental formula for multiplication [25, 37, 38, 39]:

$$i^2 = j^2 = k^2 = ijk = -1 \quad (3.10)$$

iii. The History of Octonions

"If with your alchemy you can make three pounds of gold, why should you stop there?" asked John T. Graves in a letter in which he was replying to Hamilton, who happened to be his dear friend from college, congratulating him on the birth of his brilliant new idea of quadruplets. On December 26th of the same year, Graves wrote to his friend about an eight-dimensional norm division algebra, which he named "Octaves." Hamilton did not publish his friend's work at the time. Consequently, young British mathematician Arthur Cayley (1821-1895), who showed his interest in Hamilton's theory of quaternions since the announcement of their existence, published a paper that included the same idea of Grave's octonions in March 1845, and they became known as "The Cayley Numbers" [7, 39, 40, 41].

IV. Constructing The Hyper – Complex Numbers

i. Quaternions**:**

Since the complex numbers were constructed $z = a + bi$ in a form of "duel or double" system. We likely are going to consider the form:

$$z = a + bi + cj \quad (4.1)$$

Where $a, b, c \in \mathbb{R}$ and $\boldsymbol{i}$ and $\boldsymbol{j}$ are certain symbols [9]. It is noticeable from the complex numbers that we could adopt the following addition rule:

$$(a_1 + b_1 i + c_1 j) + (a_2 + b_2 i + c_2 j) = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j \quad (4.2)$$

Thinking about the rule of multiplication will guide us to a ground-breaking conclusion. Let's start with a simple example to build the foundation of our work.

$$(a + 0i + 0j)(b + 0i + 0j) = ab + 0i + 0j \quad (4.3)$$

Which states that the multiplication for number under $\mathbb{R}$ holds [9]**.** This rule implies that:

1. the product of the number $k = k + 0i + 0j$ and by a number $z = a + bi + cj$ must equals $kz = ka + kbi + kcj$
2. The equality holds for some numbers z_1, z_2 and some arbitrary real numbers a, b as follows:
$$(az_1)(bz_2) = (ab)(z_1 z_2) \quad (4.4)$$

Furthermore, the laws of distribution, commutativity, and associativity [9]. Nonetheless, the satisfaction of these laws does not imply the probability of having unrestricted division system for the whole system [9]. As example, we cannot divide 1 by $\boldsymbol{i}$ as illustrated in the following equation:

$$(0 + 1i + 0j)x = 1 + 0i + 0j \quad (4.5)$$

The previous equation has no solution. This idea is not a random coincidence. There is a possibility to show that the previous equation satisfies the multiplication rules. Even though it is impossible to make a division system out of the number $z = a + bi + cj$ [9].

That is a major conclusion that led the Irish scientist William Rowan Hamilton in the year 1843 to solve the problem of the inability to create a division system by introducing the quaternions [9, 18, 19].

$$q = a + bi + cj + dk \quad (4.6)$$

The set of the quaternions can be written in the following form [28]:

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\} \quad (4.7)$$

The quaternions are categorized as an associative algebra with 1 as the multiplicative unit [45].

ii. Octonions

The octonions were discovered by the Irish mathematician John T. Graves, a friend of William Rowan Hamilton in the year 1843 in order to generalize the study of quaternions and extend its ideas [44]. To construct the octonions, we would likely use John C. Baez [7] method to construct them. We will conduct the construction of octonions by showing their multiplication table. The octonions are a division algebra with an 8-dimensional algebra that have 8 bases $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ [7]. Their multiplication is described by a multiplication table, which elucidate the product of multiplying the *i*th row by the *j*th column. The following table 1 illustrates the product of all permutations of an octonion's 8 factors [7, 44]:

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table 1 : Multiplication table of octonions

We deduce from the previous table that [7]:

- $e_1, \ldots, e_7$ is a root of -1
- e_i and e_j are anticommute when $i \neq j$

These major conclusions will help us to define the octonions. Octonions are a generalization of the quaternions to a higher dimensional lever, where the octonions are in the form of [44, 45]:

$$a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 =$$

$$(a_0,\ a_1, a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7) = (a_0, \vec{a}) \quad (4.8)$$

Where $a \in \mathbb{R}$. Therefore, the set of the octonions can be written in the form [44, 45]:

$$\mathbb{O} = \left\{ a_0 + \sum_{i=1}^{7} a_i e_i : a_1, \ldots, a_7 \in \mathbb{R} \right\} \quad (4.9)$$

The octonions are categorized as a non-associative algebra with 1 as the multiplicative unit [45].

V. Algebraic Operations, Multiplications Diagrams, and Mathematical Definitions

i. Quaternions

The quaternions have a basic addition rule, similar to the dual system of complex numbers [9, 42, 44]:

$$(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k)$$

$$= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k \quad (5.1)$$

Despite that the quaternions have a basic addition rule, they have a unique multiplication rule. To determine the multiplication algorithm, the way to multiply *i*, *j*, and *k* needs to be known a way by using the following diagram 1 represents the multiplication diagram for quaternions roots [7, 9]:

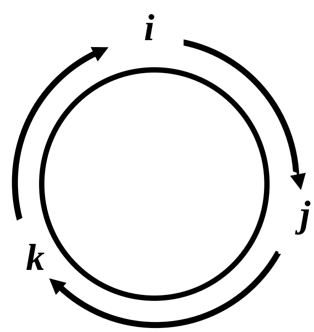

Figure 1: The multiplication diagram for the quaternions

Given this time that [7, 9]:

$$i^2 = -1,\ j^2 = -1,\ k^2 = -1,\ ijk = -1 \quad (5.2)$$

$$ij = k,\ jk = i,\ ki = j \quad (5.3)$$

$$ji = -k,\ ik = -j,\ kj = -i \quad (5.4)$$

The previous diagram is the same as a multiplication table, where the three components of the number are arranged clockwise around the circle. The product of two components results in the third component or the negative component according to the direction of the multiplication [7, 9, 46].

The rules of multiplication are called Hamilton's Rules. The set of quaternions was denoted by $\mathbb{H}$ in honor of Hamilton's numbers discovery. The following figure 2, represents the orthogonal state of a quaternions and it can be illustrated mathematically through the Hamilton's Rules [44, 46]:

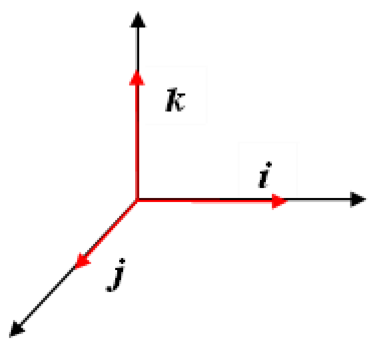

Figure 2 : A quaternion in an orthogonal state

After introducing the multiplication diagram for the quaternions, we can multiply two arbitrary quaternions. Thus, Let

$$q_1 = (a_1 + b_1 i + c_1 j + d_1 k) \quad (5.5)$$

$$q_2 = (a_2 + b_2 i + c_2 j + d_2 k) \quad (5.6)$$

By using the multiplication diagram [9, 44]:

$$q_1 q_2 = a_1 a_2 + a_1(b_1 i) + a_1(c_2 j) + a_1(d_2 k) + (b_1 i)a_2 + (b_1 i)(b_2 i) + (b_1 i)(c_2 j) + (b_1 i)(d_2 k) + (c_1 j)a_2 + (c_1 j)(b_2 i) + (c_1 j)(c_2 j) + (c_1 j)(d_2 k) + (d_1 k)a_2 + (d_1 k)(b_2 i) + (d_1 k)(c_2 j) + (d_1 k)(d_2 k) \quad (5.7)$$

In spite the non-commutative nature of the quaternions, dealing with them is still possible since they are considered associative [9, 44].

$$(q_1 q_2) q_3 = q_1 (q_2 q_3) \quad (5.8)$$

After clearing the definitions, operations, and properties of quaternions, we are ready to learn more about new forms of a quaternion that most of us know. The conjugate of a quaternion is denoted as $\bar{q}$, while the absolute of a quaternion or the magnitude is denoted as $|q|$ [9].

$$\bar{q} = a - bi - cj - dk \quad (5.9)$$

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2} \quad (5.10)$$

From these equalities, we can conclude the following product [9]:

$$q \cdot \bar{q} = |q| \quad (5.11)$$

ii. Octonions

The addition operation of two octonions is identical to the complex numbers and the quaternions [7, 44, 46]. Thus, let $a, a' \in \mathbb{O}$

$$a + a' = (a + a') + (b + b')e_1 + (c + c')e_2 + (d + d')e_3 + (e + e')e_4 + (f + f')e_5 + (g + g')e_6 + (h + h')e_7 \quad (5.12)$$

The octonions are ridiculously huge to deal with. Similar to the quaternions, the octonions had a multiplication table that can be translated into a diagram. In order to conduct the process of multiplying octonions, we need to calculate all the possible products out of any permutation out of the $\{e_1, e_{2,} e_{3,} e_4, e_5, e_6, e_7\}$ [7, 44].

We are going to rely on a well-known structure in the graph theory called the Fano plane [7, 9, 44]. The Fano plan is an apparatus with 7 points and 7 lines. The "lines" are the sides of the triangle, its altitudes, and the circle containing all the midpoints of the sides. Each pair of distinct points lies on a unique line. Each line contains three points. The following figure 3 illustrates the Fano plane used to multiple octonion factors [7]:

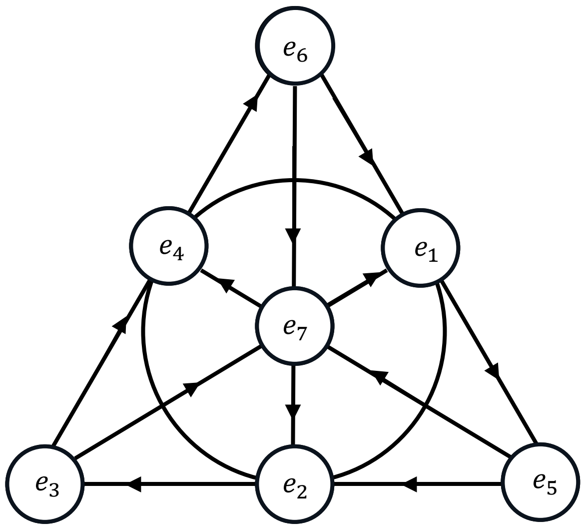

Figure 3 : The “Fano plane” for octonions multiplication

If e_i, e_j and e_k are cyclically ordered in this way, then:

$$e_i e_j = e_k \quad , \quad e_j e_i = -e_k \qquad (5.13)$$

According to the previous statement, these rules hold:

- 1 is the multiplicative identity.
- $e_1, \ldots, e_7$ are square roots of -1

The Fano plane explains the algebraic structure of the octad system of the octonions. Nevertheless, the Fano plane of octonions multiplication is not the full story. The octonions are projective structures over the 2-element field $\mathbb{Z}_2$. Precisely, they consist of lines passing through the origin in the vector space $\mathbb{Z}_2^3$ [7]. In conclusion, by assuming that $1 \in \mathbb{O}$ (the octonions multiplicative identity), then the Fano plane can be thought of as the following figure 4 which shows the visualization of the “Fano plane” by assuming that $1 \in \mathbb{O}$ [7]:

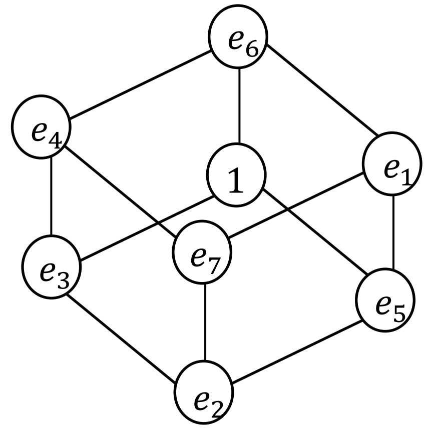

Figure 4: The visualization of the “Fano plane” by assuming that $1 \in \mathbb{O}$

After accessing the required algorithm for multiplying octonions components, we can examine the process of multiplication through an example. Let $u, v \in \mathbb{O}$, our multiplication operation can be conducted by using the vector form of the octonions and multiplying using a matrix [44]

$$u \cdot v = (u_0,\, u_1,\, u_2,\, u_3,\, u_4,\, u_5,\, u_6,\, u_7) \cdot (v_0,\, v_1,\, v_2,\, v_3,\, v_4,\, v_5,\, v_6,\, v_7) =$$

$$\begin{bmatrix} u_0v_0 & -u_1v_1 & -u_2v_2 & -u_3v_3 & -u_4v_4 & -u_5v_5 & -u_6v_6 & -u_7v_7 \\ u_1v_0 & u_0v_1 & u_2v_4 & -u_4v_2 & u_5v_6 & -u_6v_5 & u_3v_7 & -u_7v_3 \\ u_2v_0 & u_0v_2 & u_3v_5 & -u_5v_3 & u_6v_7 & -u_7v_6 & u_4v_1 & -u_1v_4 \\ u_3v_0 & u_0v_3 & u_4v_6 & -u_6v_4 & u_7v_1 & -u_1v_7 & u_5v_2 & -u_2v_5 \\ u_4v_0 & u_0v_4 & u_1v_2 & -u_2v_1 & u_5v_7 & -u_7v_5 & u_6v_3 & -u_3v_6 \\ u_5v_0 & u_0v_5 & u_2v_3 & -u_2v_3 & u_6v_1 & -u_1v_6 & u_7v_4 & -u_4v_7 \\ u_6v_0 & u_0v_6 & u_3v_4 & -u_4v_3 & u_7v_2 & -u_2v_7 & u_1v_5 & -u_5v_1 \\ u_7v_0 & u_0v_7 & u_4v_5 & -u_5v_4 & u_1v_3 & -u_3v_1 & u_2v_6 & -u_6v_2 \end{bmatrix}$$

Note: The octonions multiplication is a non-commutative operation. Moreover, the octonions multiplication is also a non-associative operation [7, 18, 44, 46]. These properties can be verified through the following example [44]:

$$(u_1 \cdot v_2) \cdot w_3 = (uv)_4 \cdot w_3 = -(uvw)_6$$

$$u_1 \cdot (v_2 \cdot w_3) = u_1 \cdot (vw)_5 = (uvw)_6$$

Where we use the notation, $u_1 = (0, u, 0, 0, 0, 0, 0, 0)$, $(uv)_2 = (0, 0, uv, 0, 0, 0, 0, 0)$ [and so on…] for the octonions containing only one non-zero element [44].

VI. Cayley – Dickson Construction

The Cayley – Dickson construction is an algebraic construction that relate normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [7]. This construction proposes a sort of a pattern that generates a sequence of infinite algebras relating each algebra with the other. Cayley – Dickson construction contribute to the interpretation of the non–commutativity of quaternions $\mathbb{H}$ and the non–associativity of octonions $\mathbb{O}$. This outstanding construction was named after the mathematicians Arthur Cayley and Leonard Dickson [9].

As Hamilton has noted, the complex numbers in form of $z = a + bi$ can be thought of as an ordered pair in the form of (a, b) where $a, b \in \mathbb{R}$ [7, 47]. The addition operation is done with respect to respective components and the multiplication operation is as follows [7, 47]:

$$(a, b)(c, d) = (ac - bd, ad + bc) \quad (6.1)$$

A conjugate of a complex number can be represented in the following form [7]:

$$(a, b)^* = (a, -b) \quad (6.2)$$

After constructing the complex numbers from the real numbers, we can execute the same methodology with the quaternions. The quaternions can be thought of as an ordered pair of complex numbers. As always, the addition is done component – wise, and multiplication is as follows [7]:

$$(a, b)(c, d) = (ac - db^*, a^*d + cb) \quad (6.3)$$

The conjugate of the quaternions can be represented as:

$$(a, b)^* = (a^*, -b) \quad (6.4)$$

And there is a pattern to a sequence of hypercomplex numbers. The octonions can be defined as a pair of quaternions. Furthermore, the addition and multiplication are defined with the same formulas. This idea of an algebra emerging from another algebra is called the Cayley – Dickson construction [7, 9, 47].

The real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ all have multiplicative inverses [7, 9]. The idea of a multiplicative inverse can be concluded from the following operation between a complex number and its conjugation [7]:

$$(a, b)(a, b)^* = (a, b)^*(a, b) = k(1, 0) \quad , \quad k \in \mathbb{R} \quad (6.4)$$

The same idea holds for quaternions and octonions. As we know, the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are all considered division algebras. Nevertheless, there isn't an infinite sequence of division algebras. By using the Cayley – Dickson construction we can get the algebra following the octonions to the infinity, but our resulting algebra turns to be worse than previous one. First, we lose the order, then we lose the commutativity, then we lose the associativity, and finally we lose the property of the division algebra [7].

By continuously applying the Cayley – Dickson construction to the octonions, we get a sequence of algebras of dimensions 16, 32, 64, and so on. The first formed algebra after the octonions is called the sedenions (a 16 – dimensional number system) [7, 48]. The sedenions are not real, non – commutative, and neither associative nor alternative. However, the sedenions are not a division algebra, and hence all the following algebras have zero divisors [7, 49, 50].

VII. QQM (Quaternion Quantum Mechanics)

Quantum mechanics is a foundational theory in modern physics that aims to describing physical phenomena and properties of nature on an atomic – quantum – scale. Many scientists along the years tried to find the correct interpretation of the quantum mechanics theory as it might guide us to the ability to fully describe the behavior of our universe.

The quaternion quantum mechanics QQM represented a significant benefaction that might answer the central question of quantum mechanics interpretation. The quaternion quantum mechanics was proposed for the first time in the year 1936 by Birkhoff and J. von Neumann [51, 52].

Quaternions are denoted as $\mathbb{H}$, where the quaternion notation formulated by Hamilton is that a quaternion is a sum of a real scalar and an imaginary vector part [51]: $\sigma = \sigma_0 + \hat{\phi} = [\sigma_0 + \hat{\phi}] \in \mathbb{H}$. A quaternion $\sigma \in \mathbb{H}$ can be written as [51]:

$$\sigma = (\sigma_0 + \phi_1 + \phi_2 + \phi_3) \in \mathbb{H} \quad (7.1)$$

Where σ_0, $\phi_i \in \mathbb{R}$. Let $\mathbb{R}^4$ be the four-dimensional Euclidean vector space with the orthonormal basis $\{e_0, e_1, e_2, e_3\}$, such that $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$, $e_3 = (0,0,0,1)$ with a three – dimensional vector subspace $P = span\{e_1, e_2, e_3\}$ [51]. The multiplication formula is as follows [51]:

$$a \cdot b = \left(a_0 b_0 - \hat{a} \circ \hat{b}\right) e_0 + \hat{a} \times \hat{b} + a_0 \hat{b} + b_0 \hat{a} \quad (7.2)$$

Where $a = \sum_{i=0}^{3} a_i e_i$, $b = \sum_{i=0}^{3} b_i e_i \in \mathbb{R}^4$, $\hat{a} = \sum_{i=0}^{3} a_i e_i$, $\hat{b} = \sum_{i=0}^{3} b_i e_i \in P$ and $\circ$, $\times$ means scalar and vector products in P [51]. Then we can deduce that [51]:

$$\hat{a} \circ \hat{b} = \sum_{i=0}^{3} a_i b_i \quad (7.3)$$

$$\hat{a} \times \hat{b} = det \begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \quad (7.4)$$

Let $\Omega \subset \mathbb{R}^3$ be a bounded set. The $\mathbb{H}$ – valued function can be written as:

$$\sigma(x) = \sigma_0(x) + \phi_1(x) i + \phi_2(x) j + \phi_3(x) k \quad , \quad x = (x_1, x_2, x_3) \in \Omega \quad (7.5)$$

Where the functions $\sigma_0(x)$, and $\phi_i(x)$ are real – valued functions. Continuity, differentiability, integrability and so on are assigned to σ must be possessed by the four components $\sigma_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$. Then, the Banach, Hilbert and Sobolev spaces of $\mathbb{H}$ – valued functions can be defined [51, 53]. In the Hilbert space over $\mathbb{H}$,

$$L^2(\Omega) = \left\{ \sigma : \Omega \to \mathbb{H} \mid \int_{\Omega} \sigma_0^2 \, dx < \infty , \int_{\Omega} \phi_i^2 \, dx < \infty , i = \{1,2,3\} \right\} \quad (7.6)$$

We define the Sobolev spaces,

$$H^k(\Omega) = \left\{ \sigma : \Omega \to \mathbb{H} \mid \sigma, \sigma^{(1)}, \ldots, \sigma^{(k)} \in L^2(\Omega) \right\}, k \in \mathbb{N} \quad (7.7)$$

Similarly, the functions $\sigma(t, x)$ depending on time t may be considered. The operator Cauchy – Riemann D will be acting on the quaternion – valued function as follow:

$$D\sigma(t, x) = \left(-div \, \hat{\phi}\right) 1 + grad \, \sigma_0 + rot \, \hat{\phi} \, , \sigma = \sigma_0 1 + \hat{\phi} \quad (7.8)$$

Where $grad \, \sigma_0 = \frac{\partial \sigma_0}{\partial x_1} i + \frac{\partial \sigma_0}{\partial x_2} j + \frac{\partial \sigma_0}{\partial x_3} k$, $div \, \hat{\phi} = \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_3}{\partial x_3}$ and

$$rot \hat{\phi} = det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$$

Under the restriction $div \hat{\phi} = 0$, D corresponds to the nabla operator ∇ in $\mathbb{R}^3$:

$$D\sigma(t, x) = grad \, \sigma_0 + rot \, \hat{\phi} \, , \sigma = \sigma_0 1 + \hat{\phi} \quad (7.9)$$

Where σ is a $\mathbb{H}$ – valued function.

Note: $DD\sigma = -\Delta\sigma$, thus equation (7.9) links quaternion quantum mechanics to reality in $\mathbb{R}^3$

After stating the required fundamentals to work with the quaternions, we can start to link the quaternions with the reality. Deformation fields represents the vector field representation when a force is applied to an object, they are either compression (irrotational) or twist (rotational). The compression field is denoted by $\sigma_0 = div \, u$ and twist field is denoted by $\hat{\phi} = rot \, u$. Helmholtz made a use of quaternions by proposing the Helmholtz decomposition, furthermore he proved that any deformation field u can be decomposed to a compression field u_0 and a twist field u_ϕ [51, 53]. Hence,

$$u = u_0 + u_\phi \ , \sigma_0 = div\, u_0 \ , \ \hat{\phi} = rot\, u_\phi \quad (7.10)$$

In the year 1822 Cauchy finished his theory of the ideal elastic continuum or in other words the Cauchy displacement mechanics [51, 54, 55]. Cauchy's displacement mechanics was specified for calculating the mechanical behavior of elastic body. Cauchy developed an equation called Cauchy equation of motion in order to describe the elastic bodies mathematically. The equation of motion relates the acceleration u due to the displacement with the variables of field deformation: compression, and twist [51, 53].

$$\frac{\partial^2 u}{\partial t^2} = 3c^2 graddiv\, u - c^2 rotrot\, u \quad (7.11)$$

Where $c = \sqrt{0.4\Upsilon/\rho p}$ such that Υ is young's modulus and ρp is continuum density. The previous equation (7.11) means that the acceleration equals the twist of the twist subtracted from the gradient of the compression. The equation represented a huge conundrum as it cannot be reduced to a vectorial model [51, 56]. After many tries Hamilton realized that the problem cannot be modelled in algebra $\mathbb{R}^3$ vector space. Moreover, he realized it needs a 4 – dimensional vector space. Therefore, a deformation field σ can be written as a quaternion such that the compression σ_0 is the scalar (real part) and the twist $\hat{\phi}$ is the vector (imaginary part).

$$\begin{bmatrix} Mechanical \\ potential \end{bmatrix} = [Compression] + [Twist] \implies$$

$$[quaternion] = [scalar] + [vector]$$

$$[\sigma] = [\sigma_0] + [\phi_1 i + \phi_2 j + \phi_3 k] \quad (7.12)$$

By combining Hamilton quaternion algebra [51] and Cauchy's classical mechanics [51, 55, 56], it starts to relate with the quaternion quantum mechanics. By combining Cauchy equation of motion with Helmholtz decomposition of fields equation and applying the divergence to it we get,

$$div\left(\frac{\partial^2}{\partial t^2}(u_0 + u_\phi) = 3c^2 graddiv(u_0 + u_\phi) - c^2 rotrot\, u\right) =$$

$$\frac{\partial^2}{\partial t^2}(div\, u_0 + div\, u_\phi) = 3c^2 divgrad(div\, u_0 + div\, u_\phi)$$

Let's substitute by $divrot\, A = 0,\ div\, u_\phi = 0,$ $\sigma_0 = div\, u_0$.

$$\frac{1}{3c^2}\frac{\partial^2 \sigma_0}{\partial t^2} = \Delta\sigma_0 \quad (7.13)$$

The previous equation (7.13) represents a longitudinal wave in $\mathbb{R}^3$ [51].

By combining Cauchy equation of motion with Helmholtz decomposition of fields equation and applying the rotation to it we get,

$$rot\left(\frac{\partial^2}{\partial t^2}(u_0 + u_\phi) = 3c^2 graddiv(u_0 + u_\phi) - c^2 rotrot\, u\right),$$

$$rot\, u_0 = 0$$

$$\frac{\partial^2}{\partial t^2}(rot\, u_0 + rot\, u_\phi) = c^2 graddiv(rot\, u_0 + rot\, u_\phi)$$

$$+2c^2 graddiv(rot\, u_0 + rot\, u_\phi) - c^2 rotrot(rot\, u_0 + rot\, u_\phi) =$$

$$\frac{\partial^2}{\partial t^2} rot\, u_\phi = c^2 graddiv(rot\, u_\phi) - c^2 rotrot(rot\, u_\phi) + 2c^2 rot[graddiv(u_0 + u_\phi)]$$

By substituting $\vec{\phi} = rot\, u_\phi$. We get,

$$\frac{\partial^2 \vec{\phi}}{\partial t^2} = c^2\left(\overrightarrow{graddiv\, \phi} - \overrightarrow{rotrot\, \phi}\right) = c^2 \Delta\vec{\phi}$$

Then we replace $graddiv\, A - rotrot\, A = \Delta A$

$$\frac{\partial^2 \vec{\phi}}{\partial t^2} = c^2 \Delta\vec{\phi} \quad (7.14)$$

The previous equation (7.14) represents a transverse wave [51]. Therefore, we can conclude that by combining Cauchy equation of motion with Helmholtz decomposition of fields equation, we can form many shapes of waves.

$$\frac{\partial^2 \sigma}{\partial t^2} = c^2 \Delta\sigma_0 + 2c^2 \Delta u_0 \quad (7.15)$$

We formulated a general second – order partial differential equation that will be used generously in the following examples. The energy of deformation field per unit mass is represented by the following equation [51]:

$$e = \frac{1}{2}\hat{u}\cdot\hat{u}^* + \frac{1}{2}c^2\sigma\cdot\sigma^* + c^2\sigma_0^2 \quad (7.16)$$

e = energy per mass unit in the deformation field

$\sigma = \sigma_0 + \hat{\sigma}$, $\sigma^* = \sigma_0 - \hat{\sigma}$

$\hat{u} = \frac{\partial u}{\partial t}$

Stationary wave ≡ particle m in Ω [51]:

$$E_m(\Omega) = \int_\Omega \rho p\left(\frac{1}{2}\hat{u}\cdot\hat{u}^* + \frac{1}{2}c^2\sigma\cdot\sigma^* + c^2\sigma_0^2 + c^2\tilde{V}(x)\right)dx \quad (7.17)$$

$E_m(\Omega)$ = total energy in the deformation field

$\tilde{V}(x)$ = external field

By substituting $\psi = \sqrt{\frac{\rho p}{(2m)}}\sigma$ in the equation (7.17) of the total energy. We get,

$$E_m(\Omega) = mc^2\int_\Omega \left(\frac{m_P}{m}\frac{\rho p}{2m}\left(\frac{\hat{u}}{c}\cdot\frac{\hat{u}^*}{c}\right) + \psi\cdot\psi^* + \frac{2m}{m_P c^2}V(x)\psi\cdot\psi^*\right)dx$$

Let's use the Cauchy – Riemann operator D such that $\underbrace{\frac{\hat{u}}{c}}_{normalized\ velocity} = \underbrace{-l_P D\sigma}_{normalized\ gradiet\ of\ mechnical\ potential}$

$$E_m(\Omega) = mc^2\int_\Omega \rho p\left(\frac{m_P}{m}\frac{\rho p}{2m}(D\sigma\cdot D\sigma^*) + \psi\cdot\psi^* + \frac{2m}{m_P c^2}V(x)\psi\cdot\psi^*\right)dx$$

Then by minimizing the expression we get The Du Bois Reymond lemma [51, 57].

$$-\frac{m_p^2c^2l_p^2}{2m}\Delta\psi + V(x)\psi = \lambda\psi \quad (7.18)$$

where a constant factor on the right-hand side can be considered as extra energy of the particle in the presence of the field $V = V(x)$. It has to be satisfied with the condition $div\,\hat{\psi} = 0$ where $\psi = \psi_0 + \hat{\psi}$. Finally, we end up with the invariant Schrödinger equation:

$$-\frac{\hbar^2}{2m}\Delta\psi + (V(x) - \lambda)\psi = 0 \quad (7.19)$$

We fully formulated the Schrödinger equation from quaternions in the equation (7.19). By Similar approach to the complex – time dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi \quad (7.20)$$

We can introduce the quaternion form:

$$\frac{1}{3}(i+j+k)\hbar\frac{\partial\psi}{\partial t} == -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi \quad (7.21)$$

Let's substitute the function $\Psi(t,x) = e^{-(i+j+k)\frac{E}{\hbar}t}\psi(x)$ in the equation (7.21). In fact, by substituting this arbitrary function in the equation we will get time – dependent Schrödinger equation (7.20).

$$\Psi(t,x) = \left[\cos\left(\sqrt{3}\frac{E}{\hbar}t\right) - \frac{1}{\sqrt{3}}(i+j+k)\sin\left(\sqrt{3}\frac{E}{\hbar}t\right)\right]\psi(x),$$

$$\frac{\partial\psi}{\partial t}(t,x) = \left[-\sqrt{3}\frac{E}{\hbar}\sin\left(\sqrt{3}\frac{E}{\hbar}t\right) - \frac{1}{\sqrt{3}}(i+j+k)\frac{E}{\hbar}\cos\left(\sqrt{3}\frac{E}{\hbar}t\right)\right]\psi(x),$$

$$\frac{\partial\psi}{\partial t}(t,x) = -(i+j+k)\frac{E}{\hbar}\left[\cos\left(\sqrt{3}\frac{E}{\hbar}t\right) - \frac{1}{\sqrt{3}}(i+j+k)\sin\left(\sqrt{3}\frac{E}{\hbar}t\right)\right]\psi(x) =$$

$$-(i+j+k)\frac{E}{\hbar}\,e^{-(i+j+k)\frac{E}{\hbar}t}\psi(x) \quad (2.22)$$

Obviously,

$$\Delta\Psi(t,x) = e^{-(i+j+k)\frac{E}{\hbar}t}\Delta\psi(x) \quad (7.23)$$

Then it can be concluded that that equation (7.23) implies the equation (7.20) [51]. Think about the case where $\Psi_1 = \Psi_2 = \Psi_3$ and let $\widetilde{\Psi} := \Psi_1 = \Psi_2 = \Psi_3$. Then $\Psi \coloneqq \Psi_0 + \frac{i+j+k}{\sqrt{3}}\widetilde{\Psi}$ solves the quaternion time – dependent Schrödinger equation $\Leftrightarrow \Psi \coloneqq \Psi_0 + i\widetilde{\Psi}$ solves the complex Schrödinger equation.

$$\frac{1}{\sqrt{3}} i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi \quad (7.24)$$

In conclusion, we have discussed the quaternions, Cauchy equations of motion, Helmholtz decomposition and quaternion quantum mechanics. We gathered the required information to relate between the quaternions with the elasticity models of Cauchy and derive from these models the well – known foundations of quantum mechanics.

VIII. Three-Dimensional Rotation

One of the main applications of quaternions is the three-dimensional rotation that describes the attitude of a rigid body. Before getting to using quaternions to represent three-dimensional rotation, we will briefly explore other approaches.

i. Euler Angles

Euler angles is a common method to describe orientation as a sequence of three rotations about three mutually perpendicular axes. To do so, a widely used method is the "heading-pitch-bank" system that performs the rotation according to the following steps:

1- Start with the original orientation.
2- Heading: Perform the rotation with angle θ about the y-axis.
3- Pitch: Measures the amount of rotation ψ about the object-space x-axis or the angle of declination.
4- Bank: Measures the amount of rotation φ about the object-space z-axis.

This process gives the possibility of forming up to 12 different sequences of rotation expanded as follows:

$$\begin{matrix} xyz & yzx & zxy \\ xzy & yxz & zyx \\ xyx & yzy & zxz \\ xzx & yxy & zyz \end{matrix}$$

To ensure the uniqueness of each orientation using Euler angles, the heading angle θ and the bank angle φ are restricted to a domain of [-180°,180°], whereas the pitch angle ψ (The second rotation) is restricted to a domain of [-90°,90°]. As much as it seems easy to perform rotations with Euler angles, an irritating problem might be encountered in certain cases. If we set the patch angle to ±90°, it may force the first and the third rotations (heading and bank) to be performed about the same axis or to be aligned. This phenomenon is known as the *Gimbal Lock*, illustrated in figure 5.

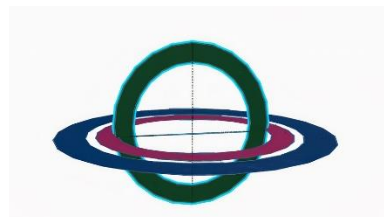

Figure 5: The Gimbal Lock

To avoid this problem, the heading rotation can be performed about the vertical axis and the bank angle is set to 0°. However, this method is still possible to represent any 3-D rotation with Euler angles [58, 59, 60].

ii. The Axis-Angle Representation

Euler's rotation theorem states that three-dimensional rotation can be accomplished via one rotation about one axis instead of 3 [61]. Hence, an angular displacement can be described according to 2 values: an angle of rotation θ and a unit vector n̂: (θ, n̂). Since n̂ is a unit vector with a norm value of 1, we can multiply it by θ without causing any troubles. Consequently, we can form what is called an *exponential map*. Such that,

$$e = \theta\hat{n} \quad (8.1)$$

and $\theta = \|e\|$ [5].

iii. Quaternion 3-D Rotation

As previously mentioned, the set of quaternions define the elements in $\mathbb{R}^4$, but an alternative representation of quaternions is defining them by two parts: a scalar (real) part and a vector part in $\mathbb{R}^3$. By this we can represent a quaternion q as

$$q = q_0 + q = q_0 + iq_1 + jq_2 + kq_3 \quad (8.2)$$

where q_0 is the scalar and q is the 3-D vector. A quaternion with a q_0 value of zero is called a Pure Quaternion. Thus, the product of a vector and a quaternion is the same as the quaternion product of a quaternion and a pure quaternion [62]. According to Euler's theorem of rotation, the rotation of a 3-D vector occurs about an axis of rotation **u** and an angle of rotation θ. Thus, for a unit quaternion

$$\mathrm{q} = q_0 + \mathrm{q} = \cos\frac{\theta}{2} + \mathrm{u}\sin\frac{\theta}{2} \quad (8.3)$$

A vector **v** in $\mathbb{R}^3$ can be written as $v = a + n$, where a is the component along q and n is normal to q. For any unit quaternion, an operator on a vector **v** in $\mathbb{R}^3$ can be defined using a unit quaternion as follows:

$$L_q(v) = qv\bar{q} =$$

$$(q_0^2 - \|q\|^2)\mathrm{v} + 2(\mathrm{q}\cdot\mathrm{v})\mathrm{q} + 2q_0(\mathrm{q}\times\mathrm{v})$$

$$(8.4)$$

Where $\bar{q}$ is the quaternion conjugate and $\|q\|$ is the norm. The operator L_q does not change the length nor the direction of **v**. In other words, a is invariant and n is the rotation about q with angle θ. And since L_q is in fact a linear operator, $qv\bar{q}$ can be considered a rotation of **v** about q with angle θ.

Applying the operator on the n component, we find the following:

$$L_q(\mathrm{n}) = (q_0^2 - \|q\|^2)\mathrm{n} + 2(\mathrm{q}\cdot\mathrm{n})\mathrm{q} + 2q_0(\mathrm{q}\times\mathrm{n})$$

$$= (q_0^2 - \|q\|^2)\mathrm{n} + 2q_0(\mathrm{q}\times\mathrm{n})$$

$$= (q_0^2 - \|q\|^2)\mathrm{n} + 2q_0\|q\|(\mathrm{u}\times\mathrm{n})$$

Where $u = \frac{q}{\|q\|}$. Since $n_\perp = \mathrm{u}\times\mathrm{n}$, we can rewrite the equation as

$$(q_0^2 - \|q\|^2)\mathrm{n} + 2q_0\|q\|n_\perp \quad (8.5)$$

Since $n_\perp$ has the same length as n

$$\|n_\perp\| = \|n \times u\| = \|n\|\cdot\|u\|\sin\frac{\pi}{2} = \|n\|$$

We can rewrite the equation as

$$L_q(\mathrm{n}) = \left(cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2}\right)n + \left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)n_\perp$$
$$= \cos\theta\, n + \sin\theta\, n_\perp \quad (8.6)$$

This rotation of n can be represented with the unit quaternion by substituting in (8.3) [63]:

$$L_q(\mathrm{v}) = \left(cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2}\right)\mathrm{v} + 2\left(u\sin\frac{\theta}{2}\cdot v\right)u\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\left(u\sin\frac{\theta}{2}\times v\right) = \cos\theta\cdot\mathrm{v} + (1-\cos\theta)(\mathrm{u}\cdot\mathrm{v})\mathrm{u} + \sin\theta\cdot(\mathrm{u}\times\mathrm{v}) \quad (8.7)$$

The same can be applied to any rotation of a vector in $\mathbb{R}^3$to be represented with a unit quaternion. Representation of 3-D rotation with a unit quaternion is preferable in multiple fields, especially in game development, 3-D graphics, and robotics since it offers the advantages of continuity and ease of construction compared to other approaches such as the rotation matrices [64, 65].

IX. Conclusion

After the preceding investigation, we can conclude that the study of the higher-dimensional complex numbers is a vital field of mathematics, specifically abstract algebra, engaging in various applications and areas of study. There are four known norm division algebras: Real Numbers $\mathbb{R}$, Complex Numbers $\mathbb{C}$, Quaternions $\mathbb{H}$, and Octonions $\mathbb{O}$ with dimensions 1,2,4, and 8, respectively. The discovery of complex numbers went through hundreds of years between acceptance and disapproval, from unveiling the existence of the square root of -1 to Gauss's construction of the two-dimensional complex plane. In 1843, Hamilton crowned his intensive work on complex numbers and their generalization with his discovery of quaternions: Associative, non-commutative under multiplication four-dimensional algebras with the imaginary units i, j and k. Hamilton's rules of multiplication: $i^2 = j^2 = k^2 = ijk = -1$. In the same year, John T. Graves generalized the study of Hamilton by extending his Quaternions to eight dimensions, constructing the Octonions: Non-associative, non-commutative under multiplication eight-dimensional algebras of the form $\mathbb{O} = \{a_0 + \sum_{i=1}^{7} a_i e_i : a_1, \ldots, a_7 \in \mathbb{R}\}$, where $e = \sqrt{-1}$. The Cayley-Dickson Construction is a method developed by mathematicians Arthur Cayley and Leonard Dickson used to obtain new algebras from old algebras by defining the new algebra as a product of an algebra with itself by conjugation.

Consequently, this construction gives us the reasons why octonions are larger than quaternions and quaternions can fit into the set of octonions, and so with complex and real numbers. Additionally, it tells us why $\mathbb{H}$ is non-commutative under multiplication and why $\mathbb{O}$ is non-associative. Working with imaginary numbers in such ways may seem ambiguous and ridiculous. Hyper-complex numbers are crucial to quantum mechanics since they might be the key to find the correct interpretation of quantum mechanics theory. Furthermore, quaternion rotation forms the fundamentals of kinematic modeling in robots, and octonions are essential in other branches of abstract algebra. Thus, this paper is an insight into the world of insane imaginary numbers with a fascinating demonstrated ability to be applied physically in the real world.

X. References

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